

Some topics in complex and harmonic analysis, 5

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Let $f(x)$, $\phi(x)$ be continuous functions on \mathbf{R}^n , and assume for simplicity that f is bounded and that ϕ is integrable. For instance, ϕ might satisfy a bound like

$$(1) \quad |\phi(x)| \leq \frac{C}{1 + |x|^{n+1}}$$

for some nonnegative real number C and all $x \in \mathbf{R}^n$. In this event we can define the convolution of f and ϕ in the usual manner,

$$(2) \quad (f * \phi)(x) = \int_{\mathbf{R}^n} f(y) \phi(x - y).$$

This is especially simple if ϕ has compact support, in which case f can be any continuous function on \mathbf{R}^n .

Let us make the normalizing assumption that the integral of ϕ on \mathbf{R}^n is equal to 1. For each positive real number t define $\phi_t(x)$ by

$$(3) \quad \phi_t(x) = t^{-n} \phi(t^{-1} x).$$

Thus ϕ_t is a continuous integrable function on \mathbf{R}^n for all $t > 0$ whose integral is also equal to 1. For small t ϕ_t is basically mostly concentrated near 0, while for t large ϕ_t is more diffuse. In particular, for each $r > 0$ the integral of ϕ_t on the ball with center 0 and radius r tends to 1 as $t \rightarrow 0$.

Using the continuity of f one can check that $(\phi_t * f)(x)$ tends to $f(x)$ as $t \rightarrow 0$ for all $x \in \mathbf{R}^n$. Basically, $(f * \phi_t)(x)$ is an average of f which is mostly concentrated around x as $t \rightarrow 0$, and thus it tends to $f(x)$ as $t \rightarrow 0$. Because f is uniformly continuous on compact subsets of \mathbf{R}^n , one can show that $f * \phi_t$ converges to f uniformly on compact subsets of \mathbf{R}^n as $t \rightarrow 0$. If f is uniformly continuous, then $f * \phi_t$ converges to f uniformly on \mathbf{R}^n . There are analogous statements for unbounded functions f under suitable conditions on ϕ .

In some cases it may be that the limit of $f * \phi_t(x)$ as $t \rightarrow 0$ exists even though f is not continuous at x . As a basic scenario, suppose that $n = 1$, and that the right and left limits of f exist at a point x . Suppose also that ϕ is an even function, which is to say that $\phi(-x) = \phi(x)$. In this case one can check that $(f * \phi_t)(x)$ tends to the average of the left and right limits of f at x as $t \rightarrow 0$.

Let us continue to suppose that $n = 1$, and consider the case where ϕ is a rational function on the real line. In other words, $\phi(x)$ can be written as $p(x)/q(x)$, where p, q are polynomials and q does not vanish on the real line. In order for ϕ to be integrable we should assume that the degree of q is at least the degree of p plus 2.

Notice that ϕ_t is then a rational function as well for all $t > 0$. Let us assume for simplicity that f has compact support in the real line. The convolution $f * \phi_t$ is an integral of translates of the rational function of ϕ , and we can approximate it by finite sums of translates of ϕ_t , using Riemann sums. Of course finite sums of translates of a rational function are again rational functions.

Thus we get a nice way to approximate f by rational functions, namely by approximating f by $f * \phi_t$ and then approximating the convolution by a finite sum of translates of ϕ_t . A rational function is in particular analytic, which means that it has a convergent Taylor series expansion in the neighborhood of any point. The size of the neighborhood may be quite small, because the poles of the rational function may be near by.

Fix a closed and bounded interval $[a, b]$ in the real line, and suppose that $r(x)$ is a rational function with no poles on this interval. Using partial fractions we can write $r(x)$ as a linear combination of polynomials and rational functions of the form $(x + c)^{-l}$, where c is a complex number not in the interval $[a, b]$ and l is a positive integer. Each of these building blocks has a convergent Taylor series expansion about some point in the complex plane which converges uniformly on the interval $[a, b]$. In this way one can see that a rational function can be uniformly approximated by polynomials on any closed and bounded interval in the real line on which it does not have a pole. One can use this and the earlier arguments to show Weierstrass' approximation theorem, to the effect that a continuous function on a closed and bounded interval in the real line can be approximated uniformly by polynomials.

In any dimension one can choose ϕ to be continuously differentiable of all orders and to have compact support, so that if f is a continuous function on \mathbf{R}^n , then $f * \phi_t$ is continuously differentiable of all orders and converges to f uniformly on compact subsets of \mathbf{R}^n , assuming that the integral of ϕ is equal to 1. One might prefer to choose ϕ to be real-analytic, so that ϕ has a convergent Taylor expansion in a neighborhood of any point. In this case the support of ϕ will be all of \mathbf{R}^n , since otherwise ϕ would be identically equal to 0. One can still choose ϕ to have enough decay to be integrable, so that $f * \phi_t$ is defined for a suitable class of functions f and converges to f uniformly on compact subsets of \mathbf{R}^n .

Another interesting class of functions ϕ to consider on \mathbf{R}^n are functions of the form

$$(4) \quad \phi(x) = \theta_1(x_1) \theta_2(x_2) \cdots \theta_n(x_n),$$

where the θ_j 's, $1 \leq j \leq n$, are continuous integrable functions on the real line with integral equal to 1. Again ϕ_t would have the same form. If the θ_j 's have compact support and f is a continuous function on \mathbf{R}^n with compact support, then $f * \phi_t$ converges uniformly to f on \mathbf{R}^n and has support contained in a

fixed compact set when $t \leq 1$, say, and $f * \phi_t$ can be approximated by finite sums of products of functions of one variable with supports contained in a fixed compact set, because of the form of ϕ . In short we get a nice way to approximate f uniformly by finite sums of products of functions of one variable with supports contained in a fixed compact set.

One might instead wish to choose functions ϕ which are radial, which is to say that ϕ can be written as $\rho(|x|)$, where ρ is a continuous function of one variable. This is the same as saying that ϕ is invariant under orthogonal linear transformations on \mathbf{R}^n , and of course ϕ_t is a radial function for all $t > 0$ when ϕ is radial. In this situation the mapping which sends a function f to $f * \phi_t$, under suitable integrability conditions, has the nice feature that it commutes with orthogonal linear transformations on \mathbf{R}^n . In other words, if R is an orthogonal linear transformation on \mathbf{R}^n , and one first sends f to the composition $f \circ R$, and then convolves the result with ϕ_t , then that is the same as taking $f * \phi_t$ and composing it with R . In particular, $f * \phi_t$ is a radial function if f is radial.

References

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